▨ The 1-D Wave Equation

- Start w/ the wave equation in 1-d

$$\frac{d^2y}{dx^2} - \frac{1}{v^2}\frac{d^2y}{dt^2} = 0$$

- We are already used to the idea of normal modes as good "building blocks" for more complicated ways that, say, a string can vibrate.
- The normal modes look like a function of x, which describes the shape of the mode, times a function of t that describes its oscillation.
- So let's look for sol'ns of this eqn that look like $f(x)\,g(t)$. Plug this into eqn:

$$g(t)\,f''(x) - \frac{1}{v^2} f(x)\,\ddot{g}(t) = 0$$

← dot for d/dt
prime for d/dx

- Now <u>divide</u> by $f(x)g(t)$:

$$\rightarrow \underbrace{\frac{1}{f(x)} f''(x)}_{\text{FUNCTION OF } x} - \underbrace{\frac{1}{v^2}\frac{1}{g(t)}\ddot{g}(t)}_{\text{FUNCTION OF } t} = 0$$

- Something interesting has happened. The variables x & t are independent. So how can a function of x & a function of t add up to zero for <u>all</u> values of x & t? If it worked @ x = 12 cm & t = 5s, then it probably wouldn't work @ t = 9s, right?
- There's only one way this can work if x & t are really independent.

- Each part of the eqn must be equal to a constant, and the constants must cancel:

$$\left.\begin{aligned} \frac{1}{f(x)} f''(x) &= \text{CONSTANT} \\ \frac{1}{g(t)} \ddot{g}(t) &= v^2 \times (\text{SAME CONSTANT}) \end{aligned}\right\} \Rightarrow \frac{f''}{f} - \frac{1}{v^2}\frac{\ddot{g}}{g} = 0$$

- Let's assume our constant is <u>negative</u> (we'll see why later) and write it as '$-k^2$'. Then:

$$\underbrace{f''(x) = -k^2 f(x) \qquad \ddot{g}(t) = -k^2 v^2 g(t)}$$

This is remarkable! We made a simplifying guess about a sol'n of the PDE, and it turned into two <u>ODEs</u>!

- We know how to solve these equations!

$$f(x) = A\cos(kx) + B\sin(kx)$$

$$g(t) = C\cos(kvt) + D\sin(kvt)$$

- So we've found a family of solutions of the wave eqn in 1-D. This is called a "separable" sol'n b/c it can be expressed as the product of functions of each independent variable.

$$f(x)\,g(t) = \big(A\cos(kx) + B\sin(kx)\big) \times \big(C\cos(kvt) + D\sin(kvt)\big)$$

- This is more complicated than the normal modes we're used to: there are 5 arbitrary unknowns here (A, B, C, D, & k).

- But that makes sense – differential eqns have multiple sol'ns & we have to specify things like boundary conditions & initial conditions (one for each derivative) to find a <u>specific</u> solution!

- Also, remember that the wave eqn is <u>LINEAR</u>. So if I add two sol'ns w/ different k_1 & k_2 the result is still a sol'n. Or I could add together a million of them, or even an ∞ number of sol'ns.

- The separable sol'n we just worked out is a general building block. We will add them in specific combinations (using what we've learned about Fourier series!) to build more complicated sol'ns.

- So, given a set of boundary & initial conditions, our strategy will be to impose as many of them as possible @ the level of separable sol'ns. Then we'll add up those sol'ns & figure out what must be done to impose any remaining conditions.

- For instance, our familiar example is a string w/ length L that is clamped (held in place) @ $x=0$ & $x=L$. In terms of bndy conditions that means:

$$y(0,t) = y(L,t) = 0$$

Can we make our separable sol'n satisfy these boundary conditions?

(i) $f(0) = A \cancelto{1}{\cos(k\cdot 0)} + B \cancelto{0}{\sin(k\cdot 0)} = 0$

$\Rightarrow A = 0$

(ii) $f(L) = 0 \cdot \cos(kL) + B\sin(kL) = 0$

Two possibilities:

- $B = 0$, but then $A=0$ & $B=0 \Rightarrow f(x) = 0$!
- $\sin(kL) = 0 \Rightarrow k = \frac{n\pi}{L}$ w/ $n = 1, 2, 3, \ldots$

- So the general separable sol'n that satisfies the boundary conditions $y = 0$ @ $x=0$ & $x = L$ is:

$$f(x)\,g(t) = B\sin\left(\frac{n\pi x}{L}\right) \times \left(C\cos\left(\frac{n\pi v}{L}t\right) + D\sin\left(\frac{n\pi v}{L}t\right)\right)$$

And really, we can 'absorb' B into C & D (since only the combos BC & BD show up) so a better way to write this is:

$$f(x)\,g(t) = \sin\left(\frac{n\pi x}{L}\right) \times \left[C_n\cos\left(\frac{n\pi v}{L}t\right) + D_n\sin\left(\frac{n\pi v}{L}t\right)\right]$$

$k = \frac{n\pi}{L}$, so our 'building blocks' are labelled by an integer $n = 1, 2, 3, \ldots$

We may want to add diff. building blocks together. So we'd probably have diff. C & D for each n.

- Now what about initial conditions? For example, we could specify the shape of the string @ $t=0$. Maybe it has the plucked shape from our example when we were working on Fourier series.

$$\underbrace{f(x)\,g(0) = C_n\sin\left(\frac{n\pi x}{L}\right)}$$

Our sep. sol'ns all look like sine waves!

- All our sep. sol'ns look like sine waves, so if our initial condition for the shape of the string @ $t=0$ looks like anything else we're out of luck.

- At this point, we have to ask "what is the most general sol'n we can build w/ these blocks?"

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \times \left[C_n \cos\left(\frac{n\pi v}{L}t\right) + D_n \sin\left(\frac{n\pi v}{L}t\right)\right]$$

Sol'n $\forall$ $n=1,2,3,\ldots$ so add them up. The eqn is linear, so this is still a sol'n. AND, each mode is zero @ $x=0$ & $x=L$ so their sum is also zero @ $x=0$ & $x=L$!

- Now we have a lot more flexibility when imposing our initial conditions! Can we find a sol'n st the shape of the string @ $t=0$ is some $F(x)$?

$$y(x,0) = \underbrace{\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = F(x)}_{\text{WHAT IS THE FOURIER SINE SERIES OF } F(x)\text{?}}$$

Some arbitrary shape (w/ $F(0)=F(L)=0$!)

- We know how to do this! Fourier's trick gives us the C_n!

$$C_n = \frac{2}{L}\int_0^L dx\ F(x) \sin\left(\frac{n\pi x}{L}\right)$$

- But what about the D_n? If we assume that we release the string from rest @ $t=0$ then

$$0 = \dot{y}(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \times \left[-C_n \cdot \frac{n\pi v}{L} \cancelto{0}{\sin\left(\frac{n\pi v}{L}\cdot 0\right)} + D_n \cdot \frac{n\pi v}{L} \cdot \cancelto{1}{\cos\left(\frac{n\pi v}{L}\cdot 0\right)}\right]$$

$$\Rightarrow\ 0 = \sum_{n=1}^{\infty} D_n \cdot \frac{n\pi v}{L} \sin\left(\frac{n\pi x}{L}\right) \quad \Rightarrow\ D_n = 0$$

- More generally, if the string is <u>moving</u> in the y-dir. @ $t=0$ then we might describe its velocity @ each pt. x by $\dot{y}(x,0) = V(x)$. Then:

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} D_n \cdot \frac{n\pi v}{L} \cdot \sin\left(\frac{n\pi x}{L}\right) = V(x)$$

$$\Rightarrow D_n = \frac{2}{L} \cdot \frac{L}{n\pi v} \cdot \int_0^L dx\, V(x) \sin\left(\frac{n\pi x}{L}\right)$$

- And that's it! Starting from our general separable sol'n we applied the B.C. & then the initial conditions to find a unique sol'n.

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \times \left[C_n \cos\left(\frac{n\pi v}{L} t\right) + D_n \sin\left(\frac{n\pi v}{L} t\right)\right] \\ C_n &= \frac{2}{L}\int_0^L dx\, F(x) \sin\left(\frac{n\pi x}{L}\right) \\ D_n &= \frac{2}{n\pi v}\int_0^L dx\, V(x) \sin\left(\frac{n\pi x}{L}\right) \end{aligned}}$$

Sum of building blocks that satisfy $y=0$ @ $x=0, x=L$.

Shape of string @ $t=0$ is $F(x)$. Velocity of string @ pt. x @ $t=0$ is $V(x)$

This is the general sol'n of the wave equation for a <u>clamped</u> string.

- Of course, we'd get a different sol'n w/ different bndy conditions! The pressure wave in a tube (like in a pipe organ) satisfies the same eqn. If the tube is closed @ one end & open @ another the BC are:

$$p(0,t) = 0 \quad \Rightarrow \quad A = 0$$

$$p'(L,t) = 0 \quad \Rightarrow \quad B \cdot k \cdot \underbrace{\cos(kL)}_{0 \text{ if } kL = \pi/2,\, 3\pi/2,\, 5\pi/2,\, \dots} = 0 \quad \Rightarrow \quad k = \frac{(2n+1)\pi}{2L}$$

p has an <u>anti-node</u> @ $x = L$

- But back to our string example. Why did we use a negative number $(-k^2)$ for our S.o.V. constant? Well, what if we'd used a positive number?

$$f'' = +k^2 f \Rightarrow f = Ae^{kx} + Be^{-kx}$$
$$\ddot{g} = +v^2k^2 g \Rightarrow g = Ce^{kvt} + De^{-kvt}$$

Also a sol'n of the wave eqn!

- This is a perfectly good sol'n of the wave eqn, it just isn't that useful for our clamped string problem! Look what happens when we try to enforce the $y=0$ B.C. @ $x = 0, L$:

$$f(0) = Ae^0 + Be^0 = A+B = 0 \Rightarrow B = -A$$

$$\hookrightarrow f(L) = Ae^{kL} - Ae^{-kL} = 0 \Rightarrow e^{kL} = e^{-kL}$$

$$\Rightarrow e^{2kL} = 1$$

One sol'n is $k=0$, but then $f(x) = 0$! The only other possibility is to recall

$$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow k = iK \text{ \& } 2KL = 2n\pi \Rightarrow K = \frac{n\pi}{L}$$

But then $k = i\frac{n\pi}{L} \Rightarrow k^2 = -\frac{n^2\pi^2}{L^2}$!

- The lesson here is that there may be more than one family of separable sol'ns, and we need to find the ones that are well-suited to the B.C. and initial conditions for the problem we're trying to solve!

- Let's look @ one more example of S.o.V. for a PDE w/ two independent variables:

$$\frac{du}{dt} = \alpha^2 \frac{d^2u}{dx^2} - \beta u \quad , \quad u(0,t) = u(L,t) = 0 \text{ \& } u(x,0) = F(x)$$

$\alpha, \beta > 0$!!!

- First, let's look for a separable sol'n $X(x)T(t)$:

$$X\dot{T} = \alpha^2 T X'' - \beta X T$$

DIVIDE BY XT

$$\frac{\dot{T}}{T} + \beta = \alpha^2 \frac{X''}{X}$$

- As before, each side must equal the <u>same</u> constant, which we'll call "P." It could be positive, zero, or negative.

$$\alpha^2 \frac{X''}{X} = P \qquad\qquad \frac{\dot{T}}{T} + \beta = P$$

- We have to enforce (or try to enforce) the BC $X(0)=0$ & $X(L)=0$, so let's just focus on $X(x)$ & the three possibilities $P>0$, $P=0$, & $P<0$.

(i) $P>0 \Rightarrow P = +k^2$

Don't <u>have</u> to do this, but writing P in a form that makes it <u>explicitly</u> positive can be helpful!

$$\alpha^2 \frac{X''}{X} = k^2 \rightarrow X'' = \frac{k^2}{\alpha^2} X \Rightarrow X(x) = A e^{\frac{k}{\alpha}x} + B e^{-\frac{k}{\alpha}x}$$

(ii) $P = 0 \Rightarrow \alpha^2 \frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X(x) = A + Bx$

(iii) $P<0 \Rightarrow P = -k^2$

$$\alpha^2 \frac{X''}{X} = -k^2 \Rightarrow X'' = -\frac{k^2}{\alpha^2} X \Rightarrow X(x) = A\cos\left(\frac{k}{\alpha}x\right) + B\sin\left(\frac{k}{\alpha}x\right)$$

- Now what happens in each case when we check $X(0)=0$ & $X(L)=0$?

(i) $P = +k^2$ $\quad X(0) = A + B = 0 \Rightarrow B = -A$

$$\rightarrow X(x) = A\times\left(e^{\frac{k}{\alpha}x} - e^{-\frac{k}{\alpha}x}\right) \rightarrow X(L) = A\times\left(e^{\frac{kL}{\alpha}} - e^{-\frac{kL}{\alpha}}\right) \Rightarrow \begin{cases} A = 0 \\ k = 0 \end{cases}$$

FORCE $X(x)=0$!

(ii) $P = 0 \Rightarrow X(x) = A + Bx$

$X(0) = A + \cancel{B0} = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$

$X(L) = B \cdot L = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$ ✗ ← No Good!

(iii) $P = -k^2 \rightarrow X(x) = A\cos\left(\frac{k}{\alpha}x\right) + B\sin\left(\frac{k}{\alpha}x\right)$

$X(0) = A \cdot \cos(0) + B \cdot \cancelto{0}{\sin(0)} = 0 \Rightarrow A = 0$

$\rightarrow X(x) = B\sin\left(\frac{k}{\alpha}x\right)$

$X(L) = B\sin\left(\frac{k}{\alpha}L\right) = 0 \Rightarrow \begin{cases} B = 0 \leftarrow \text{Forces } X(x)=0! \text{ No good} \\ \frac{k}{\alpha}L = \pi, 2\pi, 3\pi, \ldots \Rightarrow k = \frac{n\pi\alpha}{L} \quad n \in \mathbb{Z},\ n \geq 1 \end{cases}$

- So depending on the sign of P there are 3 qualitatively different possibilities for $X(x)$. But only one of them $(P<0)$ seems to be compatible w/ the B.C. we want to impose! ← Diff. B.C.? Diff. sol'ns become relevant!

$$P = -k^2 \Rightarrow X(x) = B\sin\left(\frac{n\pi}{L}x\right) \quad \text{w/ } n \in \mathbb{Z} \ \& \ n \geq 1$$

← $n=0$ gives $X=0$!

- Now we can solve the $T(t)$ eqn for this case:

$$\frac{\dot{T}}{T} + \beta = -\frac{n^2\pi^2\alpha^2}{L} \Rightarrow \dot{T} = -\left(\beta + \frac{n^2\pi^2\alpha^2}{L^2}\right)T$$

- This eqn has just one derivative, and gives an exponential:

$$T(t) = C\, e^{-\left(\beta + \frac{n^2\pi^2\alpha^2}{L^2}\right)t}$$

- So the general separable sol'n consistent with our B.C. @ $x=0$ & $x=L$ (the "NORMAL MODES" for the problem) are:

$$X(t)\,T(t) = B\sin\left(\frac{n\pi}{L}x\right) e^{-\left(\beta + \frac{n^2\pi^2\alpha^2}{L^2}\right)t} \qquad n \in \mathbb{Z},\ n \geq 1$$

B/t B in X & C in T, there's really just one overall constant

- Now we need to impose the initial condition $u(x,0) = F(x)$. Note that there's only one t-derivative so only 1 initial condition is needed.

- First, can we make the initial condition work w/ just one normal mode?

$$\underbrace{X(x)T(0) = B \sin\left(\frac{n\pi}{L}x\right) e^{\cancel{-\left(\beta+\frac{n^2\pi^2\alpha^2}{L^2}\right)\cdot 0}} \stackrel{?}{=} F(x)}_{\text{Only if } F(x) \text{ happens to be a sine wave w/ period } \frac{2L}{n}\text{!}}$$

- For anything else, a single normal mode won't work. So we use our normal modes as building blocks & look for a combination that works.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\beta+\frac{n^2\pi^2\alpha^2}{L^2}\right)t}$$

Add up all permissible building blocks. We'll determine the amount B_n of each one that we need!

$$\underbrace{u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = F(x)}_{\text{What is the Fourier sine series of } F(x)\text{?}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\beta+\frac{n^2\pi^2\alpha^2}{L^2}\right)t} \quad \text{w/ } B_n = \frac{2}{L}\int_0^L dx'\, F(x') \sin\left(\frac{n\pi x'}{L}\right)}$$

▨ OCS

- So far we've seen two examples of S.o.V. for PDEs w/ two independent variables : x & t.
- For one eqn (the wave eqn) the sep. sol'ns were products of trig functions.
- In our other example the sep. sol'ns were trig functions of x times exponentials of t.
- On the HW you'll solve an equation of the form

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \quad \longleftarrow \text{Laplace's Eqn in 2-D}$$

and find that the sep. sol'ns are trig functions of one variable times exponentials of the other.

- Other eqns include the 1-D heat/diffusion eqn

$$\frac{du}{dt} = \alpha \frac{d^2u}{dx^2} \quad \longleftarrow \text{Like last example, without the } \beta u \text{ term!}$$

and the 1-D Schrödinger eqn w/ a potential that depends on x (but not t):

Saw this in MODERN PHYSICS!

$$i\hbar \frac{d\psi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x)$$

- What other possibilities are there w/ two variables?
- Two things come to mind. First, we might want to solve Laplace's eqn in, say, polar coords or some other OCS instead of x & y.

$$\nabla^2 u(\rho,\phi) = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2u}{d\phi^2} = 0$$

- Separable sol'n? $u(\rho,\phi) = f(\rho)\,g(\phi)$

$$0 = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right)g + \frac{1}{\rho^2} f \frac{d^2 g}{d\phi^2}$$

DIVIDE BY fg, MULTIPLY BY ρ^2

$$0 = \frac{\rho}{f}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) + \frac{1}{g}\frac{d^2 g}{d\phi^2}$$

- So we have a function of ρ plus a function of ϕ. The only way these can add to zero is if they are equal to constants w/ the same magnitude but opposite sign.

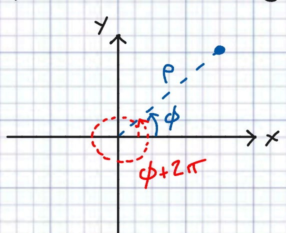
- For reasons that will become clear in a moment, let's start w/ the function $g(\phi)$. There are 3 possibilities for the constant:

(i) $\frac{1}{g}g'' = +k^2 \Rightarrow g(\phi) = A e^{k\phi} + B e^{-k\phi}$

(ii) $\frac{1}{g}g'' = 0 \Rightarrow g(\phi) = A + B\phi$

(iii) $\frac{1}{g}g'' = -k^2 \Rightarrow g(\phi) = A\cos(k\phi) + B\sin(k\phi)$

- Now, in our last example we used our B.C. to decide which sol'ns were relevant. But we haven't given any explicit B.C. here! However, recall that ϕ is an angle measured CCW from the x-axis.



Since ϕ is an angle, increasing it by 2π brings us around to the same point. So an IMPLICIT condition on our sol'n is $g(\phi+2\pi) = g(\phi)$!

- So how do we insure $\underbrace{g(\phi+2\pi) = g(\phi)}_{\text{"SINGLE – VALUED"}}$? ← like any function! Look @ each case.

(i) $A e^{k(\phi+2\pi)} + B e^{-k(\phi+2\pi)} = A e^{k\phi} + B e^{-k\phi} \quad \forall \phi$?

$\rightarrow A = B = 0$ (so $g=0$) or $k=0$ (contradicts $k^2>0$)

(ii) $A + B\cdot(\phi+2\pi) = A + B\phi \quad \forall \phi \Rightarrow B = 0$

$\rightarrow g(\phi) =$ CONSTANT

(iii) $A\cos(k\cdot(\phi+2\pi)) + B\sin(k\cdot(\phi+2\pi)) = A\cos(k\phi) + B\sin(k\phi)$

$\rightarrow A = B = 0$ ($g=0$) <u>or</u> $k \in \mathbb{Z}$

- There are two possibilities here. Either our S.o.V. constant is <u>zero</u>, in which case $g(\phi) =$ constant, or the S.o.V. is $-k^2$, in which case k must be an <u>integer</u> and $g(\phi) = A\cos(k\phi) + B\sin(k\phi)$.

- So what are the ρ sol'ns in these two cases?

(i) $\frac{\rho}{f}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = +k^2$ w/ $k \in \mathbb{Z}$ & $k \geq 1$ ← since $-k^2<0$, not $-k^2=0$!

↑ Opp sign from g eqn!

↳ $\rho\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = +k^2 f \Rightarrow \rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} - k^2 f = 0$

You can solve this via Method of Frobenius, but it's more work than necessary! Any time you have an eqn where (powers of ρ) − (# of derivatives) is the same in <u>every</u> term, try ρ^α as a sol'n!

$f(\rho) = \rho^\alpha$? $\Rightarrow \alpha(\alpha-1)\rho^\alpha + \alpha\rho^\alpha - k^2\rho^\alpha = 0$

$\rightarrow (\alpha^2 \cancel{-\alpha} \cancel{+\alpha} - k^2)\rho^\alpha = 0$

$\Rightarrow \alpha = \pm k$

So the sol'n in this case is

$$f(\rho) = C\rho^{k} + D\rho^{-k}$$

(ii) $\frac{\rho}{f}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = 0 \rightarrow \frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) = 0$ ← So the quantity inside (...) must be a <u>constant</u>

$$\rightarrow \rho\frac{df}{d\rho} = D \rightarrow \frac{df}{d\rho} = \frac{D}{\rho} \Rightarrow f(\rho) = C + D\ln\rho$$

- Now we know the <u>single-valued</u> separable sol'ns, the ones consistent w/ $u(\rho,\phi+2\pi) = u(\rho,\phi)$.

$$f(\rho)g(\phi) = \begin{cases} C + D\ln\rho & \leftarrow g(\phi) \text{ a constant, so 'absorb' into } C \,\&\, D \\ (A\cos(k\phi) + B\sin(k\phi))(C\rho^{k} + D\rho^{-k}) & k\in\mathbb{Z},\ k\geq 1 \end{cases}$$

- Without imposing any further B.C., the general single-valued sol'n is:

$$\boxed{u(\rho,\phi) = a_0 + b_0\ln\rho + \sum_{k=1}^{\infty}\left[a_k\rho^{k}\cos k\phi + b_k\rho^{-k}\cos k\phi + c_k\rho^{k}\sin k\phi + d_k\rho^{-k}\sin k\phi\right]}$$

- As w/ our previous examples, there were other separable sol'ns besides the ones included here (like $e^{\pm k\phi}$). But they weren't consistent w/ our <u>IMPLICIT</u> condition that the function should be single-valued since $\phi + 2\pi$ & ϕ are supposed to be the same angle.

An implicit condition is something we often don't bother to state explicitly b/c it seems obvious, but it has important consequences for narrowing down the sol'ns we consider.

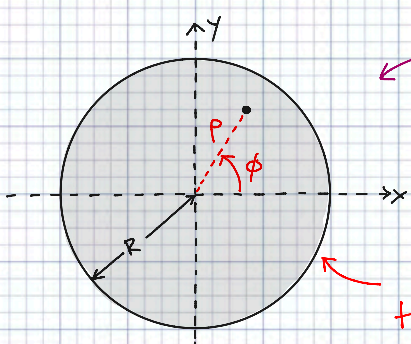
- At this point, if we had boundary conditions we could impose them & try to narrow down the sol'n.

- But first, let's look @ another example of an <u>implicit</u> condition on our sol'n.

- Our ρ sol'ns take the form ρ^k, ρ^{-k}, and $\ln\rho$. If we're looking @ the whole x-y plane, then $0 \le \rho < \infty$.

- Why is this important? B/c two of our sol'ns — ρ^{-k} & $\ln\rho$ — behave badly @ $\rho = 0$! And ρ^k & $\ln\rho$ are badly behaved @ $\rho \to \infty$! If we want a sol'n that applies @ $\rho = 0$ or $\rho \to \infty$, we'll need to address this!

- For example, suppose I want to find a sol'n of Laplace's eqn ($\nabla^2 u = 0$) on a disk of radius R, and I know that $u(R, \phi)$ is some known function — we'll call it $\sigma(\phi)$.



It's a <u>disk</u>, so it makes sense to use <u>POLAR</u> <u>COORDS</u> w/ $\rho = 0$ (the origin) @ the center of the disk.

Is $u(\rho, \phi)$ a quantity that we think should be <u>FINITE</u>? If so, then $b_0 = 0$, $b_k = 0$, & $d_k = 0$ $\forall k$, b/c $\ln\rho$ & ρ^{-k} <u>diverge</u> @ $\rho = 0$!

- So a <u>finite</u> sol'n on the disk looks like

$$u(\rho,\phi) = a_0 + \sum_{k=1}^{\infty} \rho^k \left(a_k \cos(k\phi) + c_k \sin(k\phi) \right)$$

We've dropped the $\ln\rho$ & ρ^{-k} terms, so all that's left are terms that remain finite for all $0 \leq \rho \leq R$.

- Now we use what we know about $u(R,\phi)$ to pin down the a_k & c_k

$$u(R,\phi) = \sigma(\phi) = a_0 + \sum_{k=1}^{\infty} R^k \left(a_k \cos(k\phi) + c_k \sin(k\phi) \right)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} d\phi \, \sigma(\phi)$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} d\phi \, \sigma(\phi) \cos(k\phi)$$

$$c_k = \frac{1}{\pi}\int_0^{2\pi} d\phi \, \sigma(\phi) \sin(k\phi)$$

Fourier coefficients for the function $\sigma(\phi)$ that tells us about the behavior of $u(\rho,\phi)$ @ $\rho = R$.

- For example, if the B.C. was

$$u(R,\phi) = \sigma(\phi) = 1 + \cos^4(\phi) + \sin^3(\phi)$$

Then we could evaluate the integrals (or use multi-angle formulas like $\sin^3\phi = \frac{3}{4}\sin\phi - \frac{1}{4}\sin 3\phi$) to obtain:

$$u(\rho,\phi) = \frac{11}{8} + \frac{3}{4}\rho\sin\phi + \frac{1}{2}\rho^2\cos(2\phi) - \frac{1}{4}\rho^3\sin(3\phi) + \frac{1}{8}\rho^4\cos(4\phi).$$

- Likewise, if we wanted to solve $\nabla^2 u = 0$ on the region $R \leq \rho < \infty$, we'd rule out the $\ln\rho$ & ρ^k terms!

- The important point here is that S.o.V. in an OCS besides Cartesian coords may:

  1) Lead to qualitatively different sol'ns!

  $$\rho^{\pm k} = (x^2+y^2)^{\pm k/2}$$

  $$\cos(k\phi) = \cos\left(k \tan^{-1}\left(\tfrac{y}{x}\right)\right) \quad \text{etc.}$$

  Diff. than S.o.V. sol'ns of $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

  2) Require "implicit" conditions that restrict the separable sol'ns you are interested in. Examples of this are the requirement that $g(\phi + 2\pi) = g(\phi)$, or eliminating terms like $\ln\rho$ & ρ^{-k} if we're looking for a sol'n that should be finite @ $\rho = 0$.

- Before moving on, let's look @ one more example in polar coords that highlights an important property of the <u>LINEAR</u> PDEs we've been considering.

- A boundary condition like $y(0,t) = 0$ or $u(R,\phi) = 0$ is called <u>HOMOGENEOUS</u>. If two sol'ns both satisfy a homogeneous B.C. then so does their sum — that's why any combination of our Normal Modes for the clamped string gives a sol'n of the wave eqn that still satisfies the B.C. $y(0,t) = 0$.

- A B.C. like $y(0,t) = 4$ is INHOMOGENEOUS. If $y_1(0,t) = y_2(0,t) = 4$, the sum $c_1 y_1 + c_2 y_2 = 4(c_1 + c_2)$ @ $x=0$.

- Now, suppose I ask you to solve $\nabla^2 u = 0$ on the annulus $R_i \le \rho \le R_o$ $(0 < R_i < R_o)$ with the inhomogeneous B.C. $u(R_i, \phi) = \sigma_i(\phi)$ and $u(R_o, \phi) = \sigma_o(\phi)$.

- You can either proceed directly, or you can solve 2 related problems & combine them using <u>SUPERPOSITION</u>:

  (1) $\nabla^2 u_1 = 0$ w/ $u_1(R_i, \phi) = \sigma_i(\phi)$ & $u_1(R_o, \phi) = 0$

  (2) $\nabla^2 u_2 = 0$ w/ $u_2(R_i, \phi) = 0$ & $u_2(R_o, \phi) = \sigma_o(\phi)$

- Since $\nabla^2 u = 0$ is linear, $u_1 + u_2$ is also a sol'n. And the homogeneous B.C. on u_1 @ R_o & u_2 @ R_i means

$$u(R_i, \phi) = u_1(R_i, \phi) + u_2(R_i, \phi) = \sigma_i(\phi) + 0 = \sigma_i(\phi)$$

$$u(R_o, \phi) = u_1(R_o, \phi) + u_2(R_o, \phi) = 0 + \sigma_o(\phi) = \sigma_o(\phi)$$

- So we can also "snap together" sol'ns of simpler problems to get sol'ns of more complicated problems.

- For instance, let's say $R_i = 1/2$ & $R_o = 1$, and $\sigma_i(\phi) = \cos\phi$ & $\sigma_o(\phi) = \sin\phi$.

(1) $$u_1(\rho, \phi) = a_0 + b_0 \ln\rho + \sum_{k=1}^{\infty} \left[a_k \rho^k \cos k\phi + b_k \rho^{-k} \cos k\phi + c_k \rho^k \sin k\phi + d_k \rho^{-k} \sin k\phi \right]$$

$$\left.\begin{aligned} u_1(1/2, \phi) &= \cos\phi \\ u_1(1, \phi) &= 0 \end{aligned}\right\} \xrightarrow{\text{FOURIER}} \left.\begin{aligned} a_1 \cdot \tfrac{1}{2} + b_1 \left(\tfrac{1}{2}\right)^{-1} &= 1 \\ a_1 \cdot 1 + b_1 \cdot 1 &= 0 \end{aligned}\right\} \begin{aligned} a_1 &= -2/3 \\ b_1 &= +2/3 \end{aligned}$$

Other $a, b, c, d = 0$

$$\rightarrow u_1(\rho,\phi) = -\frac{2}{3}\rho\cos\phi + \frac{2}{3}\frac{1}{\rho}\cos\phi$$

(2) $$u_2(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{k=1}^{\infty}\left[a_k\rho^k\cos k\phi + b_k\rho^{-k}\cos k\phi + c_k\rho^k\sin k\phi + d_k\rho^{-k}\sin k\phi\right]$$

$$\left.\begin{aligned} u_1(1/2,\phi) &= 0 \\ u_1(1,\phi) &= \sin\phi \end{aligned}\right\} \xrightarrow{\text{FOURIER}} \left.\begin{aligned} c_1\cdot\tfrac{1}{2} + d_1\cdot\left(\tfrac{1}{2}\right)^{-1} &= 0 \\ c_1\cdot 1 + d_1\cdot 1 &= 1 \end{aligned}\right\} \begin{aligned} c_1 &= 4/3 \\ d_1 &= -1/3 \end{aligned}$$

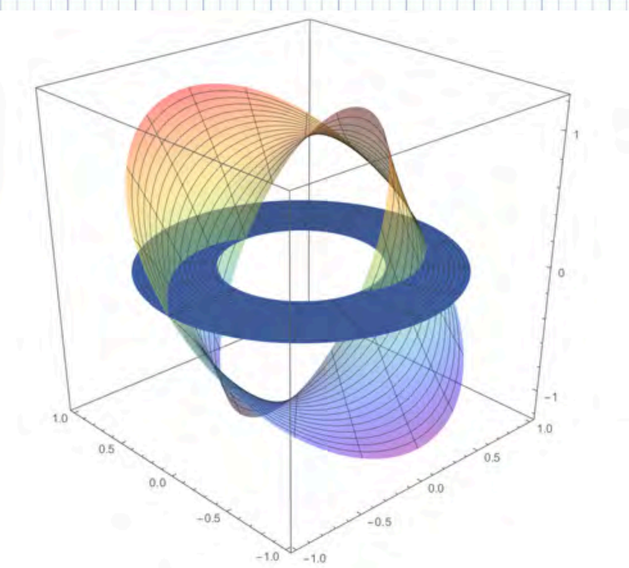
Other $a,b,c,d = 0$

$$\rightarrow u_2(\rho,\phi) = \frac{4}{3}\rho\sin\phi - \frac{1}{3}\frac{1}{\rho}\sin\phi$$

– So the sol'n of the full problem – $\nabla^2 u = 0$ on $1/2 \le \rho \le 1$ st $u(\frac{1}{2},\phi) = \cos\phi$ & $u(1,\phi) = \sin\phi$ – is:

$$u(\rho,\phi) = \left(-\frac{2}{3}\rho + \frac{2}{3\rho}\right)\cos\phi + \left(\frac{4}{3}\rho - \frac{1}{3\rho}\right)\sin\phi$$

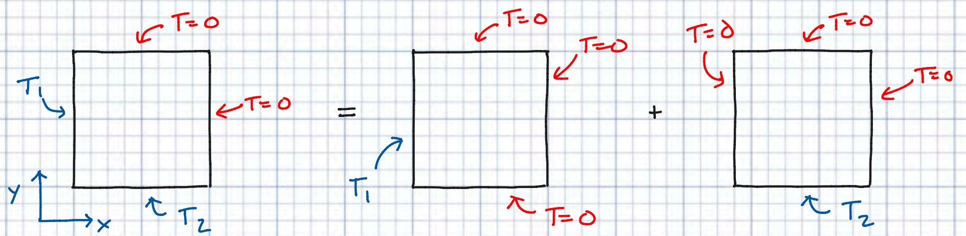
No constant, $\ln\rho$, or $\rho^{\pm k}$ w/ $k \geq 2$ terms b/c no const. or $\cos(k\phi)$, $\sin(k\phi)$ terms needed for B.C.!



The function u is equal to $\cos\phi$ on the inner boundary & $\sin\phi$ on the outer bndy. It satisfies $\nabla^2 u = 0$ $\forall$ $\frac{1}{2} \le \rho \le 1$ & $0 \le \phi < 2\pi$.

- Superposition is useful when you can break down a problem w/ N inhomogeneous B.C. into, say, N familiar problems w/ just 1 inhom. B.C.

- For instance, what is the steady-state temperature in a square plate (sides of length L) w/ $T(0,y) = T_1$, $T(x,0) = T_2$, and $T(x,L) = T(L,y) = 0$?



WHEN 3 = 2

- So what are the other possibilities for PDEs w/ 2 variables?

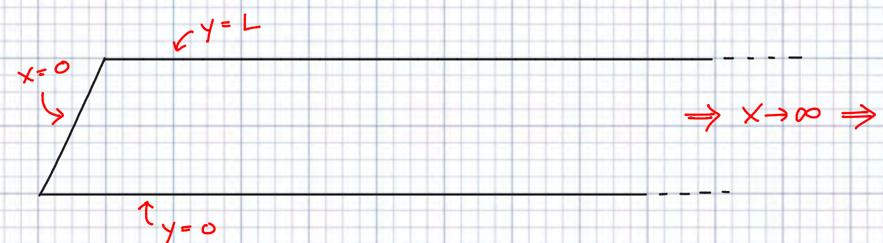
- One that comes up quite a lot is when you need 3 or more variables to describe a physical system, but you are only interested in sol'ns that depend on 2 of the variables.

- Why would that be the case? Consider the "2-D" diffusion (or heat) equation:

$$\frac{du}{dt} + \alpha^2\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right) = 0$$

- This eqn governs the diffusion of some quantity (the concentration of a chemical, say) over time throughout a region described by the Cartesian coords x & y.

- Suppose I have a very long strip w/ width W. By "very long" I mean that it's length is so much bigger than W that for our purposes it may as well be infinite.



- Now let's say we keep the concentration of the chemical <u>fixed</u> @ $x=0$, so $u(0,y,t)$ is always equal to a constant value that we'll call u_0. And we'll say the concentration along the sides ($y=0$ & $y=L$) is always zero.